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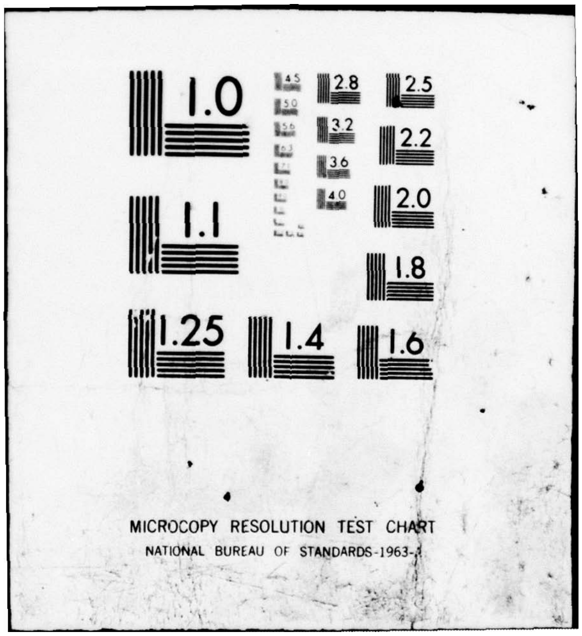
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ABSTRACT

A generalization of Freund's bivariate exponential model is discussed. Probabilistic properties of this model with minimal distribution assumptions are derived including the joint survival function and Laplace-Stieltjes transform.

1. Introduction. A study is made of a two component failure system. Let the components be denoted by A and B and their lifetimes by S and T respectively. The general model to be discussed in this paper is a generalization of that of Freund [2] and arose from consideration of bivariate exponential distributions. In addition to Freund's there have been many such distributions suggested in the literature, including those suggested by Gumbel [3], Downton [1] and Hawkes [4]. Marshall and Olkin [5] have proposed the most widely referenced model. In their model, the $\Pr\{S>s+\Delta \mid S>s, T>s\} = \Pr\{S>s+\Delta \mid S>s\}$. This implies that, conditioned on the fact that component A is functioning at time s, the distribution of its residual lifetime is independent of whether component B has failed or not.

Freund [2] derives his distribution from the assumption that at the failure of one component the distribution of the residual lifetime of the other component is changed. Specifically, he assumes that the lifetime of A given that no failure of B occurs is distributed exponentially with parameter α, denoted exp (α), and that the lifetime of B given no failure of A is distributed as exp (β). If B fails before A, then the residual lifetime of A is distributed as exp (α'). If A fails before B, then the residual lifetime of B is distributed as exp (β'). This can be summarized as follows. Let X,Y,U,V be independent random variables such that $X \sim \exp(\alpha)$, $Y \sim \exp(\beta)$, $U \sim \exp(\alpha')$ and $V \sim \exp(\beta')$. Then

$$S = \begin{cases} X & \text{if } X \le Y \\ Y+U & \text{if } X > Y \end{cases}, \quad T = \begin{cases} X+V & \text{if } X \le Y \\ Y & \text{if } X > Y \end{cases}.$$

The model introduced in section 2 is a generalization of Freund's model with the distribution assumptions dropped. Section 3 includes derivations of expressions for the joint survival function and marginal distributions for this model. Letting $Y \sim \exp(\beta)$, an explicit expression for the joint Laplace-Stieltjes transform of the distribtuion is found in section 4. Using this expression, moments of the distribution are calculated. In section 5, two definitions of system life are defined and the Laplace-Stieltjes transform of the system life distribution is found in each case.

2. Model Definition. Label the two components of a system A and B. The lifetimes of the two components are dependent, in that the failure of one component affects the residual lifetime of the other. Formally, let A and B have lifetimes S and T respectively. Given random variables X,Y,U,V write

$$S = \begin{cases} X & \text{if } X \le Y \\ Y+U & \text{if } X > Y \end{cases}, \quad T = \begin{cases} X+V & \text{if } X \le Y \\ Y & \text{if } X > Y \end{cases} \tag{2.1}$$

to represent the two lifetimes. The following assumptions are made on X,Y,U,V:

$$\begin{array}{rl} \text{i)} & X,Y,U,V \text{ are mutually independent,} \\ \text{ii)} & \Pr\{X > 0\} = 1,\ \Pr\{Y > 0\} = 1, \\ \text{iii)} & \Pr\{U < 0\} = 0,\ \Pr\{V < 0\} = 0, \\ \text{iv)} & X,Y \text{ are absolutely continuous.} \end{array} \tag{2.2}$$

Let $\Pr\{U = 0\}$ be denoted by p_U and $\Pr\{V = 0\}$ by p_V. The following is an example of the applicability of the model.

Let A and B be two generators that supply electricity to a hospital, each supplying a portion of the building. When one generator fails, the remaining gererator must supply power to the entire hospital. This puts added strain on the generator and thus alters its residual life.

3. The Joint Survival and Marginal Distributions. Let $\overline{F}(s,t) = \Pr\{S>s, T>t\}$. We begin by deriving an expression for $\overline{F}(s,t)$ when $s < t$. First condition on the values of X and Y. Then $\overline{F}(s,t) = \int_0^\infty \int_0^\infty \Pr\{S>s, T>t \mid X=x, Y=y\}\, dF_X(x)\, dF_Y(y)$. Now partition the region $[0,\infty) \times [0,\infty)$ into 12 subregions based on the relative sizes of x,y,s,t. The only non-zero contributions are $\Pr\{V>t-x\}$ on the region $[s,t] \times [x,\infty)$ and 1 on the region $[t,\infty) \times [t,\infty)$. Therefore when $s < t$,

$$\overline{F}(s,t) = \int_s^t \int_x^\infty \overline{F}_V(t-x)\, dF_Y(y)\, dF_X(x) + \int_t^\infty \int_t^\infty 1\, dF_X(x)\, dF_Y(y)$$

$$= \overline{F}_X(t)\, \overline{F}_Y(t) + \int_s^t \overline{F}_V(t-x)\, \overline{F}_Y(x)\, dF_X(x).$$

The case $s > t$ follows similarly and the case $s = t$ is trivial. We have therefore proved

Theorem 1.

$$\bar{F}(s,t) = \begin{cases} \bar{F}_X(t)\ \bar{F}_Y(t) + \int_s^t \bar{F}_V(t-x)\ \bar{F}_Y(x)\ dF_X(x) & \text{if } s < t \\ \bar{F}_X(s)\ \bar{F}_Y(s) & \text{if } s = t \\ \bar{F}_X(s)\ \bar{F}_Y(s) + \int_t^s \bar{F}_U(s-y)\ \bar{F}_X(y)\ dF_Y(y) & \text{if } s > t\ . \end{cases}$$

We can obtain the marginal distributions using $\bar{F}_S(s) = \bar{F}(s,0)$.

Corollary 2.

i) $$\bar{F}_S(s) = \bar{F}_X(s)\ \bar{F}_Y(s) + \int_0^s \bar{F}_U(s-y)\ \bar{F}_X(y)\ dF_Y(y),$$

ii) $$\bar{F}_T(t) = \bar{F}_X(t)\ \bar{F}_Y(t) + \int_0^t \bar{F}_V(t-x)\ \bar{F}_Y(x)\ dF_X(x)\ .$$

From the model definition we have the following representation for the variable S,

$$S = \min(X,Y) + U \cdot I_{\{X > Y\}}\ , \tag{3.1}$$

where $I_{\{X > Y\}}$ is the indicator function for the set $\{X > Y\}$. A similar representation exists for T. With this representation we see that in general

$$\begin{aligned} E(S) &= E(\min(X,Y)) + E(U)\ \Pr\{X > Y\}, \\ E(T) &= E(\min(X,Y)) + E(V)\ \Pr\{Y > X\}. \end{aligned} \tag{3.2}$$

4. <u>The Joint Laplace-Stieltjes Transform</u>. The joint Laplace-Stieltjes (L-S) transform of (S,T) is defined to be $\int_0^\infty \int_0^\infty e^{-as-bt} F(ds,dt)$, where $F(ds,dt)$ is the measure determined by the survival function $\overline{F}(s,t)$. We begin with a representation of $F(ds,dt)$.

<u>Lemma</u>.

$$F(ds,dt) = \begin{cases} \overline{F}_Y(s)\ dF_V(t-s)\ dF_X(s) & \text{if } s < t \\ \overline{F}_Y(s)\ p_V\ dF_X(s) + \overline{F}_X(s)\ p_U\ dF_Y(s) & \text{if } s = t \\ \overline{F}_X(t)\ dF_U(s-t)\ dF_Y(t) & \text{if } s > t \end{cases}$$

Proof for the case $s < t$: We must investigate how A can fail at time s and B fail at time t. If $s < t$, then $S < T$, and thus $X < Y$, so that $S = X = s$ and $T = X + V = s + V = t$. Therefore $X = s$, $Y > s$ and $V = t = s$. Since the variables are independent the result follows. QED

Using this lemma, the joint L - S transform of (S,T) can be written as

$$\begin{aligned} f^*(a,b) = & \int_0^\infty \int_{s+}^\infty e^{-as-bt}\ \overline{F}_Y(s)\ dF_V(t-s)\ dF_X(s)\ + \\ & p_V \int_0^\infty e^{-(a+b)s}\ \overline{F}_Y(s)\ dF_X(s)\ + \\ & p_U \int_0^\infty e^{-(a+b)s}\ \overline{F}_X(s)\ dF_Y(s)\ + \\ & \int_0^\infty \int_{t+}^\infty e^{-as-at}\ \overline{F}_X(t)\ dF_V(s-t)\ dF_Y(t) \end{aligned} \tag{4.1}$$

This expression can be evaluated piece by piece. We have

$$\int_0^\infty \int_{s+}^\infty e^{-as-bt}\bar{F}_Y(s)\, dF_V(t-s)\, dF_X(s)$$

$$= \int_0^\infty e^{-as}\bar{F}_Y(s) \int_{s+}^\infty e^{-bt}\, dF_V(t-s)\, dF_X(s)$$

$$= \int_0^\infty e^{-as}\, \bar{F}_X(s)\, e^{-bs} \int_{0+}^\infty e^{-bw}\, dF_V(w)\, dF_X(s)$$

$$= (f_V^*(b) - p_V) \int_0^\infty e^{-(a+b)s}\, \bar{F}_Y(s)\, dF_X(s)\ .$$

Here $f_V^*(b)$ is the L - S transform of F_V evaluated at b.

By a similar calculation

$$\int_0^\infty \int_{t+}^\infty e^{-as-bt}\, \bar{F}_X(t)\, dF_V(s-t)\, dF_Y(t)$$

$$= (f_U^*(a) - p_U) \int_0^\infty e^{-(a+b)s}\, \bar{F}_X(s)\, dF_Y(s)\ .$$

Combining these into (4.1) we get

Theorem 4.

$$f^*(a,b) = f_V^*(b) \int_0^\infty e^{-(a+b)s}\, \bar{F}_Y(s)\, dF_X(s) + f_U^*(a) \int_0^\infty e^{-(a+b)s}\, \bar{F}_X(s)\, dF_{Y}(s)\ .$$

These integrals cannot be evaluated in general, but can be for certain important special cases. In particular we have

Corollary 5.

If $Y \sim \exp(\beta)$, then

$$f^*(a,b) = f_V^*(b) f_X^*(a+b+\beta) + \frac{f_u^*(a)\beta}{a+b+\beta}\,[1-f_X^*(a+b+\beta)].$$

Corollary 6.

If $X \sim \exp(\alpha)$ and $Y \sim \exp(\beta)$, then

$$f^*(a,b) = \frac{1}{a+b+\alpha+\beta}\,[\alpha f_V^*(b)+\beta f_U^*(a)].$$

Using the expression in Corollary 6, moments of (S,T) can be found by evaluating the appropriate partial derivatives.

Theorem 7.

If $X \sim \exp(\alpha)$ and $Y \sim \exp(\beta)$, then

i) $E(S) = \frac{1}{\alpha+\beta}\,[1+\beta E(U)]$, $E(T) = \frac{1}{\alpha+\beta}\,[1+\alpha E(V)]$,

ii) $\mathrm{Var}(S) = \frac{1}{(\alpha+\beta)^2}\,[1+\beta^2\,\mathrm{Var}(U)+\alpha\beta E(U^2)]$,

$\mathrm{Var}(T) = \frac{1}{(\alpha+\beta)^2}\,[1+\alpha^2\,\mathrm{Var}(V)+\alpha\beta E(V^2)]$,

iii) $\mathrm{Cov}(S,T) = \frac{1}{(\alpha+\beta)^2}\,[1-\alpha\beta E(U)\,E(V)]$.

It can be shown that all values between +1 and -1 are possible for the correlation of S and T even in the special case considered in Theorem 7.

5. Time to System Failure. Failure in a series system and a parallel system will be discussed. Label the system life L_1 for the series system and L_2 for the parallel system. We have $L_1 = \min(S,T)$ and $L_2 = \max(S,T)$.

Since $\min(S,T) = \min(X,Y)$, $\bar{F}_{L_1}(t) = \bar{F}_x(t)\,\bar{F}_y(t)$ and the model does not really enter in. If X and Y are exponentially distributed then so is L_1.

To work with L_2, the random variable $D = L_2 - L_1$ will be considered. If we let $p = \Pr\{X > Y\}$ and $q = 1 - p$, then $D \sim pV + qU$. Since V and U are independent $f^*_D(a) = pq\, f^*_V(ap)\, f^*_U(aq)$. It follows from $L_2 = L_1 + D$ and the independence of L_1 and D that

$f^*_{L_2}(a) = pq f^*_V(ap)\, f^*_U(aq)\, f^*_{L_1}(a)$.

6. Discussion. The question of practicality arises here as it should with any model formulation. Is the class of models sufficiently rich yet simple enough to obtain meaningful results? The proposed model is based upon the assumption that the residual life of one component is dependent upon whether or not the other component has failed. This seems to be a realistic assumption in many applications.

The first meaningful results one would like are the moments of (S,T). For given distributions the Laplace-Stieltjes transform (Theorem 4) may be used to obtain the moments. This expression may be difficult to evaluate though. However, the much simpler expression in Corollaries 5 and 6 still come from rich classes of models. Next, given a sample can one obtain reasonable estimators for the parameters? In Freund's special case, simple expressions for the maximum likelihood estimates can be obtained.

Finally, does this model enable one to make decisions about how to control the system. When the components fail, can we arrive at optimal replacement policies? Here again, this is possible when using this model.

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